JOURNAL OF COMPUTATIONAL PHYSICS 68, 501–505 (1987)

Note

A Finite Difference Scheme for Solving a Non-linear Schrödinger Equation with a Linear Damping Term

Delfour, Fortin, and Payre [1] present a finite difference scheme for the solution of the non-linear Schrödinger equation

$$i\frac{\partial u}{\partial t}+ivu-\nabla^2 u+\lambda u(|u|^{p-1}+\alpha r)=0, \tag{1}$$

with

$$u(x, t=0)=\phi(x) \qquad \text{and} \qquad r=|x|.$$

The scheme is shown to give numerical solutions which are in good agreement with analytical solutions for cases in which the damping coefficient v is zero. However, when $v\neq 0$, the numerical solutions have sawteeth oscillations (with wavelengths of the order of the mesh spacing) superimposed on the smooth wave solutions. These sawteeth are caused by the incorrect discretization of the damping term in Eq. (1).

The numerical scheme presented by Delfour *et al.* is

$$\begin{aligned} &i\frac{u_j^{n+1}-u_j^n}{\Delta t}+iv\frac{|u_j^{n+1}|^2+|u_j^n|^2}{|u_j^{n+1}|^2-|u_j^n|^2}(u_j^{n+1}-u_j^n)\\ &\quad-\frac{1}{2}\left\{\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{(\Delta x)^2}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\Delta x)^2}\right\}\\ &\quad+\frac{\lambda}{p+1}\left\{\frac{|u_j^{n+1}|^{p+1}-|u_j^n|^{p+1}}{|u_j^{n+1}|^2-|u_j^n|^2}\right\}(u_j^{n+1}+u_j^n)+\frac{\lambda\alpha}{2}|j|\,\Delta x(u_j^{n+1}+u_j^n)=0. \end{aligned} \tag{2}$$

The unusual appearance of the damping (second) term and the non-linear (fourth) term is due to the attempt to force the numerical solution to obey discrete versions of two conservation laws for the wavefunction u in Eq. (1). Delfour *et al.* present the conservation laws

$$\int |u|^2\,dx=e^{-2vt}\int |\phi|^2\,dx, \tag{3}$$

and

$$\frac{1}{2}\int |\nabla u|^2\,dx+\frac{\lambda}{p+1}\int |u|^{p+1}\,dx+\frac{\lambda\alpha}{2}\int r\,|u|^2\,dx=\text{constant}. \tag{4}$$

0021-9991/87 $3.00

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This author has discovered that Eq. (4), and hence the original numerical scheme, is incorrect for $\nu \neq 0$. The correct version of the conservation law is

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\int |\nabla u|^2\,dx + \frac{\lambda}{p+1}\int |u|^{p+1}\,dx + \frac{\lambda\alpha}{2}\int r\,|u|^2\,dx\right] + 2\nu\left[\frac{1}{2}\int |\nabla u|^2\,dx + \frac{\lambda}{2}\int |u|^{p+1}\,dx + \frac{\lambda\alpha}{2}\int r\,|u|^2\,dx\right] = 0. \tag{5}$$

Note that the coefficients in front of the second and fifth integrals are unequal, so Eq. (5) cannot be integrated in time to give a true conservation equation.

It is doubtful that a finite difference scheme which solves Eq. (1) while subject to discrete versions of Eq. (3) and Eq. (5) could be developed. However, a change of variables in Eq. (1) can eliminate the troublesome damping term. The transformation

$$u(x, t) = e^{-\nu t} w(x, t), \tag{6}$$

applied to Eq. (1) yields

$$i\frac{\partial w}{\partial t} - \nabla^2 w + \lambda w(e^{-(p-1)\nu t}\,|w|^{p-1} + \alpha r) = 0, \tag{7}$$

with

$$w(x, 0) = \phi(x).$$

The explicit time dependence introduced to the non-linear term by the change of variables does not complicate the numerical solution of Eq. (7) beyond the need to calculate the value of the exponential at each time step.

The conservation laws for Eq. (7) are

$$\int |w|^2\,dx = \int |\phi|^2\,dx = \text{constant}, \tag{8}$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}\int |\nabla w|^2\,dx + e^{-(p-1)\nu t}\frac{\lambda}{p+1}\frac{\partial}{\partial t}\int |w|^{p+1}\,dx + \frac{\lambda\alpha}{2}\frac{\partial}{\partial t}\int r\,|w|^2\,dx = 0. \tag{9}$$

A finite difference scheme for the solution of Eq. (7) is

$$i\frac{w_j^{n+1} - w_j^n}{\Delta t} - \frac{1}{2}\left\{\frac{w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}}{(\Delta x)^2} + \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{(\Delta x)^2}\right\} + \frac{\lambda e^{-(p-1)\nu t^{n+1/2}}}{p+1}\left\{\frac{|w_j^{n+1}|^{p+1} - |w_j^n|^{p+1}}{|w_j^{n+1}|^2 - |w_j^n|^2}\right\}(w_j^{n+1} + w_j^n) + \frac{\lambda\alpha}{2}\,|j|\,\Delta x(w_j^{n+1} + w_j^n) = 0, \tag{10}$$

where

$$t^{n+1/2} = \left(n + \frac{1}{2}\right) \Delta t.$$

Note that this is identical with the scheme presented by Delfour *et al.* for the case of no damping ($\nu = 0$). The scheme presented in Eq. (10) is accurate to order $(\Delta t)^2$ and $(\Delta x)^2$. This scheme satisfies

$$\Delta x \sum_j |w_j^n|^2 = \Delta x \sum_j |w_j^0|^2 = \text{constant}, \tag{11}$$

and

$$\begin{aligned} &\frac{1}{2} \Delta x \sum_j \left[\left| \frac{w_{j+1}^{n+1} - w_j^{n+1}}{\Delta x} \right|^2 - \left| \frac{w_{j+1}^n - w_j^n}{\Delta x} \right|^2 \right] \\ &+ e^{-(p-1)\nu t^{n+1/2}} \frac{\lambda}{p+1} \Delta x \sum_j \left[|w_j^{n+1}|^{p+1} - |w_j^n|^{p+1} \right] \\ &+ \frac{\lambda \alpha}{2} \Delta x \sum_j |j| \, \Delta x \left[|w_j^{n+1}|^2 - |w_j^n|^2 \right] = 0, \end{aligned} \tag{12}$$

which are the discrete forms of the conservation laws given in Eq. (8) and Eq. (9).

Due to the non-linearity in Eq. (10), an iterative method similar to the method devised by Delfour *et al.* is employed for the solution. The iteration is continued until three convergence criteria are met. The first criterion is that the maximum difference between successive iterations of the wavefunction w is less than a given input parameter ε_1, i.e.,

$$\max_j |{}_k w_j^{n+1} - {}_{k-1} w_j^{n+1}| < \varepsilon_1, \tag{13}$$

where k denotes the iteration count. The second and third convergence criteria are that the conservation laws, Eq. (11) and Eq. (12), are satisfied to sufficient accuracy. The conservation laws may be rewritten as

$$f(w^{n+1}) - f(w^n) = 0, \tag{14}$$

and

$$g(w^{n+1}, t^{n+1/2}) - g(w^n, t^{n+1/2}) = 0, \tag{15}$$

where

$$f(w^n) = \Delta x \sum_j |w_j^n|^2, \tag{16}$$

and

$$g(w^n, t^{n+1/2}) = \frac{1}{2} \Delta x \sum_j \left| \frac{w_{j+1}^n - w_j^n}{\Delta x} \right|^2 + e^{-(p-1)\nu t^{n+1/2}} \frac{\lambda}{p+1} \Delta x \sum_j |w_j^n|^{p+1}$$
$$+ \frac{\lambda\alpha}{2} \Delta x \sum_j |j| \, \Delta x \, |w_j^n|^2. \tag{17}$$

The conservation laws are considered satisfied if

$$\left| \frac{f(w^{n+1}) - f(w^n)}{f(w^n)} \right| < \varepsilon_2, \tag{18}$$

and

$$\left| \frac{g(w^{n+1}, t^{n+1/2}) - g(w^n, t^{n+1/2})}{g(w^n, t^{n+1/2})} \right| < \varepsilon_3, \tag{19}$$

where ε_2 and ε_3 are input parameters giving the maximum allowed fractional error in the conservation laws.

This method has been used on the same test case for which the original scheme by Delfour *et al.* produced the sawteeth oscillations. The calculation was performed with $-30 \leqslant x \leqslant 30$, $0 \leqslant t \leqslant 6$, $\Delta x = 0.1$, $\Delta t = 0.02$, $\nu = 0.1$, $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = \varepsilon_3 = 5 \times 10^{-5}$, $p = 3$, and the initial condition

$$u(x, 0) = 1.5 \,\mathrm{sech}[1.5(x + 15)] \, e^{-2ix}. \tag{20}$$

The results are shown in Fig. 1, with the wavefunction plotted at every tenth time step. Although the original scheme by Delfour *et al.* produced oscillations with an amplitude of roughly 10% of the amplitude of the decaying soliton, this scheme is shown to give smooth solutions. Very low amplitude oscillations (not visible in Fig. 1) exist to the right of the soliton at the later times. These are due to the wave bouncing off the right space boundary and interfering with itself.

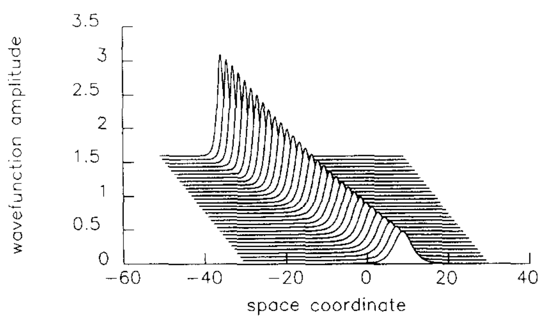

FIG. 1. Results for the damped soliton test case. (Time increases towards the viewer.)

In summary, an error has been discovered in the damping term of a scheme for the solution of a damped, non-linear Schrödinger equation presented by Delfour, Fortin, and Payre. A change of variables eliminates the damping term and allows the transformed non-linear Schrödinger equation to be solved by a scheme similar to the original scheme. The new scheme yields damped solutions which are free of the sawteeth oscillations produced by the original scheme.

Reference

1. M. Delfour, M. Fortin, and G. Payre, *J. Comput. Phys.* **44** (1981), 277.

Received: December 3, 1985; revised: March 25, 1986

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